Some semi-presentations for the sporadic almost-simple groups

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1 Introductory note

On the following two pages are semi-presentations for the 26 sporadic simple groups and their automorphism groups. A semi-presentation is a characterisation of the standard generators (in the sense of [3] and [2]) of a group using only information that can be easily found using 'black box' operations (in other words, we are only allowed to specify the orders of elements, not their conjugacy classes).

The notation '**std**' refers to the standard relations; that is, the relations that can be deduced from the definitions of standard generators given in the Web Atlas [2]. For instance, the semi-presentation:

$$\mathrm{M}_{24} \approx \langle\langle x, y \mid \mathbf{std}, o(xyxyxy^2) = 12 \rangle\rangle$$

is an abbreviation for:

$$\begin{aligned} \mathrm{M}_{24} \approx \langle\langle x, y \mid o(x) = 2, o(y) = 3, o(xy) = 23, \\ o(xyxyxy^2xyxy^2xy^2) = 4, \\ o(xyxyxy^2) = 6 \rangle\rangle \end{aligned}$$

because standard generators of M_{24} are x and y where x is in class 2B, y is in class 3A, xy has order 23 and $xyxyxyyxyxyyxyy$ has order 4.

Details of how these semi-presentations were found can be found in a preprint [1], to appear on my website shortly.

2 The sporadic simple groups

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\begin{aligned}
\mathrm{M}_{11} &\approx \langle\langle x, y \,|\, \mathbf{std} \rangle\rangle \\
\mathrm{M}_{12} &\approx \langle\langle x, y \,|\, \mathbf{std}, o(xyxyxy^2) = 6 \rangle\rangle \\
\mathrm{M}_{22} &\approx \langle\langle x, y \,|\, \mathbf{std}, o([x,y]) = 6 \rangle\rangle \\
\mathrm{M}_{23} &\approx \langle\langle x, y \,|\, \mathbf{std} \rangle\rangle \\
\mathrm{M}_{24} &\approx \langle\langle x, y \,|\, \mathbf{std}, o(xyxyxy^2) = 12 \rangle\rangle \\
\mathrm{J}_{1} &\approx \langle\langle x, y \,|\, \mathbf{std} \rangle\rangle \\
\mathrm{J}_{2} &\approx \langle\langle x, y \,|\, \mathbf{std} \rangle\rangle \\
\mathrm{J}_{3} &\approx \langle\langle x, y \,|\, \mathbf{std}, o(xyxyxy^2) = 17 \rangle\rangle \\
\mathrm{Co}_{2} &\approx \langle\langle x, y \,|\, \mathbf{std}, o(x(xy)^{14}) = 3 \rangle\rangle \\
\mathrm{Fi}_{22} &\approx \langle\langle x, y, (z) \,|\, \mathbf{std}, o(z) = 30, o(xz^{15}) = 3; z := xyxy^2xy^2 \rangle\rangle \\
\mathrm{Fi}_{23} &\approx \langle\langle x, y \,|\, \mathbf{std}, o(x^{y^2}(xy)^{14}) = 5 \rangle\rangle \\
\mathrm{Fi}'_{24} &\approx \langle\langle x, y, (z) \,|\, \mathbf{std}, o(z) = 60, o(x(z^{30})^{xyxy}) = 5; z := (xy)^6 y \rangle\rangle \\
\mathrm{HS} &\approx \langle\langle x, y \,|\, \mathbf{std}, o(xy^2) = 10, o(xyxy^2) = 15 \rangle\rangle \\
\mathrm{Suz} &\approx \langle\langle x, y \,|\, \mathbf{std}, o(xyxyxy^2) = 12 \rangle\rangle \\
\mathrm{McL} &\approx \langle\langle x, y \,|\, \mathbf{std}, o(xy^2) = 12 \rangle\rangle \\
\mathrm{He} &\approx \langle\langle x, y, (z) \,|\, \mathbf{std}, o(z) = 10, o(xz^5) = 3; z := xy^2xyxy^2xy^2 \rangle\rangle \\
\mathrm{Ru} &\approx \langle\langle x, y \,|\, \mathbf{std}, o(xy^2) = 14, o(xyxy^2) = 29 \rangle\rangle \\
\mathrm{O'N} &\approx \langle\langle x, y, (z) \,|\, \mathbf{std}, o(z) = 5, [y,z] = 1; z := xyxy(y^2(y^2)^{xyxy})^5 \rangle\rangle \\
\mathrm{HN} &\approx \langle\langle x, y \,|\, \mathbf{std}, o(x[(xy)^{11}]^{xy^2xyxyxyxy^2}) = 5 \rangle\rangle \\
\mathrm{J}_{4} &\approx \langle\langle x, y, (z,c,d,e,f,g) \,|\, \mathbf{std}, o(z) = 24, o(x(z^{12})^{xy^3xy^3}) = 11, \\
&\qquad o(g) = 20, o([g,y]) = 1; z := xyxyxy^2, c := xyxy^3xyxy, \\
&\qquad d := xy^2xy^3xy^2xy, e := c(y^2(y^2)^c)^5, \\
&\qquad f := d(y^2)(y^2)^d, g := (efe)^3(fe)^4 f \rangle\rangle \\
\mathrm{Co}_{3} &\approx \langle\langle x, y, (u,v,w) \,|\, \mathbf{std}, o(w) = 5, o([w,y]) = 1; u := (y^2(y^2)^{xy^2})^3, \\
&\qquad v := xyxy^3x^2(y^2(y^2)^{xyxy^3x^2})^2, w := (uv^2)^3(uv)^6 \rangle\rangle \\
\mathrm{Co}_{1} &\approx \langle\langle x, y, (z,a,b) \,|\, \mathbf{std}, o(z) = 42, o(x^{y^2}z^{21}) = 11, o(ab) = 36, \\
&\qquad o(ab^2abab) = 18; z := xy(xyxy^2)^2, \\
&\qquad a := (xy)^{20}, b := y^{xyxyxyxy^2} \rangle\rangle \\
\mathrm{Th} &\approx \langle\langle x, y, (z,v,w) \,|\, \mathbf{std}, o(z) = 21, o(yv^w) = 2; z := (xy)^3 y, \\
&\qquad v := z^7, w := xy^2(xy)^4(xy^2)^2(xy)^2(xy^2)^5(xy)^3 \rangle\rangle \\
\mathrm{Ly} &\approx \langle\langle x, y, (z,r,s) \,|\, \mathbf{std}, o(z) = 42, o(rs) = 14, o(rsrs^2) = 30; \\
&\qquad z := (xy)^5(xy^2)^2, r := z^{14}, s := y^{xyxy^2xyxyxy^2} \rangle\rangle \\
\mathrm{B} &\approx \langle\langle x, y, (u,v) \,|\, \mathbf{std}, o(u) = 52, o(xu^{26}) = 35, o(v) = 38, o(v^{19}y^x) = 8; \\
&\qquad u := (xyxy^2)^2(xy)^2(xyxy^2)^2; \\
&\qquad v := (xy)^3(xy^2xy)^2xy(xyxy^2)^2xy^2 \rangle\rangle \\
\mathbb{M} &\approx \langle\langle x, y, (c,d,f) \,|\, \mathbf{std}, o(c) = 50, o(xd) = 5, o(xy^f) = 34; \\
&\qquad c := (xy)^4(xy^2)^2, d := c^{25}, f := xyxyxyxyxy^2 \rangle\rangle
\end{aligned}
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3 The sporadic automorphism groups

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\begin{aligned}
\mathrm{M}_{12}.2 &\approx \langle\langle x, y \,|\, \mathbf{std}, o((xy)^3xy^2) = 6\rangle\rangle \\
\mathrm{M}_{22}.2 &\approx \langle\langle x, y \,|\, \mathbf{std}, o(xyxy^2) = 10\rangle\rangle \\
\mathrm{HS}.2 &\approx \langle\langle x, y \,|\, \mathbf{std}, o([x, y]) = 3\rangle\rangle \\
\mathrm{J}_2.2 &\approx \langle\langle x, y \,|\, \mathbf{std}, o(xy^2) = 24\rangle\rangle \\
\mathrm{J}_3.2 &\approx \langle\langle x, y \,|\, \mathbf{std}, o(xyxyxyxy^2) = 9\rangle\rangle \\
\mathrm{McL}.2 &\approx \langle\langle x, y \,|\, \mathbf{std}\rangle\rangle \\
\mathrm{Suz}.2 &\approx \langle\langle x, y \,|\, \mathbf{std}, o(xyxyxy^2xy^2) = 7\rangle\rangle \\
\mathrm{He}.2 &\approx \langle\langle x, y, (z, t) \,|\, \mathbf{std}, o(z) = 24, o(xz^{12}) = 17, o(t) = 15, o([t, y]) = 1; \\
&\qquad z := xy^2xy^2xy, t := (y^3(y^3)^x)^4((y^3(y^3)^{xy^2xy})^2\rangle\rangle \\
\mathrm{O'N}.2 &\approx \langle\langle x, y, (t, z) \,|\, \mathbf{std}, o(z) = 5, o([y, z]) = 1; \\
&\qquad z := (t(y^2(y^2)^t)^7)^2, t := xy^2xyx\rangle\rangle \\
\mathrm{Fi}_{22}.2 &\approx \langle\langle x, y, (z) \,|\, \mathbf{std}, o(z) = 22, o(xz^{11}) = 3, \\
&\qquad o((y^9)^{xy^3}(xy)^{21}) = 3; z := xyxy^5xy^4\rangle\rangle \\
\mathrm{Fi}_{24} &\approx \langle\langle x, y, (z) \,|\, \mathbf{std}, o(z) = 54, o(xz^{27}) = 3, o(xy^2) = 20; z := xyxy^6\rangle\rangle \\
\mathrm{HN}.2 &\approx \langle\langle x, y, (t, z) \,|\, \mathbf{std}, o(z) = 60, o(ty) = 22, \\
&\qquad o(ty^2(ty)^3) = 22; z := xy^3(xy)^4, t := z^{30}\rangle\rangle
\end{aligned}
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References

[1] Simon Nickerson. Semi-presentations of the sporadic simple groups. Preprint.

[2] Robert Wilson, Peter Walsh, Jonathan Tripp, Ibrahim Suleiman, Stephen Rogers, Richard Parker, Simon Norton, Simon Nickerson, Steve Linton, John Bray, and Rachel Abbott. Atlas of Finite Group Representations. Available online at `http://web.mat.bham.ac.uk/atlas/`.

[3] Robert A. Wilson. Standard generators for sporadic simple groups. *J. Algebra*, 184(2):505–515, 1996.